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COMMENT

Comment on 'Semiclassical dynamics of a spin-$\frac{1}{2}$ in an arbitrary magnetic field'

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Abstract. In a recent paper Alscher and Grabert claim to prove that a dominant (tree-level) stationary phase approximation to the *Wiener regularized* continuum $su(2)$ coherent-state path integral for the quantum propagator $\mathcal{P} := \langle z_F|T\exp(-\mathrm{i}\int_0^\tau H\,\mathrm{d}s)|z_I\rangle$ becomes exact, provided H is a linear combination of the $su(2)$ generators with arbitrary time-dependent coefficients. I find the derivation of this in fact obvious result unduly complicated and somewhat obscure. The authors start from a classical spin action inconsistent with necessary boundary conditions and therefore are forced to invoke a nontrivial regularization of the action to render the latter meaningful. Alternatively, when a $su(2)$ symplectic potential consistent with the boundary conditions is employed, no regularization is required to obtain the leading quasiclassical asymptotics.

The observation that the dominant (fluctuation-free) approximation to a path integral (PI) on a coherent-state manifold (a coadjoint orbit of a certain group G) for $\mathcal{P}$ becomes exact, provided H belongs to a Lie algebra of G, is self-evident and follows from the fact that any element of the algebra (time dependent or not) can be brought by an appropriate G-rotation to the Cartan subalgebra. This has explicitly been demonstrated for the $su(2)$ and $su(1,1)$ coherent-state PIs in [1]. Recently [2], it has also been shown that a *formal continuum* PI representation yields a correct semiclassical propagator on a group orbit up to second (Gaussian fluctuation) term, provided a symplectic one-form consistent with boundary conditions is used. The same quasiclassical form of $\mathcal{P}$ follows from a direct time-lattice evaluation of the $su(2)$ coherent-state PI [3]. In this regard, the main result of the paper [4] as it stands can hardly be considered novel.

Let us now comment on its derivation as is given in [4]. The authors start from a continuum PI representation for $\mathcal{P}$ (equation (13)) which, however, disagrees with boundary conditions imposed by fixing two points on a group orbit as the definition of $\mathcal{P}$ implies. More specifically, the $SU(2)$ symplectic one-form that defines a classical dynamics of a free spin is taken in [4] to be†

$$\theta = \frac{\mathrm{i}}{2}\frac{\bar{z}\,\mathrm{d}z - z\,\mathrm{d}\bar{z}}{1+|z|^2} = \mathrm{i}s\frac{\bar{z}\,\mathrm{d}z - z\,\mathrm{d}\bar{z}}{1+|z|^2}\Big|_{s=1/2} \qquad (1)$$

which contributes to the action

$$S_0 = \mathrm{i}\int\theta$$

† I use local complex coordinates on S^2 which are more convenient than the real ones used in [4].

with boundary conditions

$$z(0) = z_I \qquad \bar{z}(\tau) = \bar{z}_F \tag{2}$$

being understood. The Hamiltonian action principle $\delta S_0 = 0$ should yield equations of motion

$$\dot{z} = \dot{\bar{z}} = 0$$

accompanied by boundary conditions (2). But this is not the case if θ is given by (1), for nonvanishing boundary terms would emerge under variation of S_0. Therefore certain compensating terms are to be added to S_0 to ensure the correct form of the equations of motion [5]. In our case, boundary conditions (2) define two *different* (holomorphic and antiholomorphic) Lagrangian surfaces. To be consistent with the boundary conditions, θ must vanish when restricted on those surfaces, which suggests that θ is to be taken in the form

$$\theta \to \theta(\bar{z}_F, z_I) = \theta + \mathrm{i}s\left(\frac{z_I\,\mathrm{d}\bar{z}}{1+\bar{z}z_I} - \frac{\bar{z}_F\,\mathrm{d}z}{1+\bar{z}_F z}\right). \tag{3}$$

As is easily seen, equation $\delta\int\theta(\bar{z}_F, z_I) = 0$ along with boundary conditions (2) results in correct equations of motion. In view of the fact that two *different* Lagrangian surfaces are involved, a specific discontinuity of the classical trajectories at the endpoints occurs. Note also that

$$\mathrm{d}\theta = \mathrm{d}\theta(\bar{z}_F, z_I) = w^{(2)} := -2\mathrm{i}s\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^2}$$

where symplectic $SU(2)$-invariant two-form $w^{(2)}$ defines a Poisson (classical) action of the $SU(2)$ group on CP^1; quantization of the latter is possible whenever $[w^{(2)}]$ defines an integral cohomology class, which amounts to the requirement $2s \in N$.

It is natural to suggest that the basis (equation (13)) in [4] be replaced by

$$\frac{\langle z_F|T\exp-\frac{\mathrm{i}}{\hbar}\int_0^\tau H(s)\,\mathrm{d}s|z_I\rangle}{\langle z_F|z_I\rangle} = \int_{z(0)=z_I}^{\bar{z}(\tau)=\bar{z}_F}\mathrm{D}\mu_{su(2)}(z)\exp\left[\mathrm{i}\int\theta(\bar{z}_F, z_I) - \frac{\mathrm{i}}{\hbar}\int_0^\tau H^{cl}\,\mathrm{d}s\right]. \tag{4}$$

It is this representation that has been obtained in [1] for $su(2)$ coherent-state PI and generalized in [2] for a coherent-state PI quantization of rank-one group orbits. As has been shown, equation (4) is reliable as long as the leading and next-to-leading asymptotics of the quasiclassical (in powers of $1/s$) series for $\mathcal{P}$ are considered.

In order to avoid a problem with the boundary conditions, Alscher and Grabert invoke a sort of regularization of the $su(2)$ PI which is originally due to Klauder [6]. In general, an appropriate regularization of the $su(2)$ coherent-state PI is desirable, resulting in well-defined integrals over continuous paths. However, a *classical* action is meaningful without any regularization at all.

It is worth mentioning that the dominant stationary phase approximation to $\mathcal{P}$ evaluated in [4] coincides with that evaluated starting from non-regularized PI (4) (see [1]). This occurs due to the fact that Alscher and Grabert have managed, by introducing a regularization-dependent time scale, to single out a specific set of the classical paths with jumps at the endpoints, which is necessary to obtain a correct result†. Unfortunately, this trick can hardly be considered self-consistent, for the overspecification problem does not actually disappear (see equation (23)) in [4]).

† Note that the trajectories with a discontinuity at the endpoints are the only possible classical paths that naturally emerge in [1, 2].

Presumably, the approach advocated by Alscher and Grabert may be considered, provided a more accurate presentation is available, as an alternative way to evaluate the coherent-state quasiclassical propagator. It is likely that a properly regularized $su(2)$ continuum PI based on representation (4) gives rise to a quasiclassical propagator that coincides within two leading terms with that evaluated from non-regularized PI (4), but deviates from the latter at higher orders. To explicitly carry out the necessary calculations, a recently proposed 'planar' regularization of the $su(2)$ phase-space PI may turn out helpful [7].

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